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Stiff Magnetofluid Cosmological Model

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We intestigate the behavior of the magnetic field in a cosmological model filled with a stiff perfect fluid in general relativity. The magnetic field is due to an electric current along the x axis. The behavior of the model when a magnetic field is absent is also discussed.

1. INTRODUCTION

Anisotropic homogeneous universes play an important role in understanding some essential features of the universe, such as the formation of galaxies during its early stages of evolution. Jacobs (1968, 1969) investigated Bianchi type I cosmological models satisfying a barotropic equation of state in the presence of a magnetic field. Collins (1972) made a qualitative analysis of Bianchi type I models with a magnetic field. Roy and Prakash (1978) obtained a plane symmetric cosmological model with an incident magnetic field for perfect fluid distributions. Recently Bali (1986) investigated a magnetized cosmological model in which expansion (θ) in the model is proportional to σ_1^1, the eigenvalue of shear tensor σ_i^j for perfect fluid distributions. In this paper, we treat a cosmological model filled with stiff fluid in the presence of a magnetic field in general relativity. The distribution consists of an electrically neutral perfect fluid with an infinite electrical conductivity in the presence of a magnetic field.

Let us consider an anisotropic homogeneous universe in the form

$$ds^2 = A^2(dx^2 - dt^2) + B^2\, dy^2 + C^2\, dz^2 \tag{1}$$

where the metric potentials are functions of time alone. The energy-momentum tensor is taken into the form

$$T_i^j = (\varepsilon + p)v_i v^j + p g_i^j + E_i^j \tag{2}$$

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where E_i^j is the electromagnetic field given by Lichnerowicz (1967):

$$E_i^j = \bar{\mu}[|h|^2(v_i v^j + \tfrac{1}{2}g_i^j) - h_i h^j] \tag{3}$$

In the above, ε is the density, p is the presssure, and v^j is the flow vector satisfying

$$g_{ij}v^i v^j = -1 \tag{4}$$

$\bar{\mu}$ is the magnetic permeability and h_i the magnetic flux vector defined by

$$h_i = \frac{\sqrt{-g}}{2\bar{\mu}}\, \varepsilon_{ijkl}F^{kl}v^j \tag{5}$$

where F_{kl} is the electromagnetic field tensor and ε_{ijkl} the Levi-Civita tensor density. A semicolon stands for covariant differentiation. We assume that the coordinates are comoving, so that $v^1 = 0 = v^2 = v^3$ and $v^4 = 1/A$. We take the incident magnetic field to be in the direction of the x axis, so that $h_1 \neq 0$, $h_2 = 0 = h_3 = h_4$. This leads to $F_{12} = F_{13} = 0$, by virtue of (5). We also find $F_{14} = F_{24} = F_{34} = 0$ due to the assumption of the infinite conductivity of the fluid. Hence the only nonvanishing component of F_{ij} is F_{23}. The first of Maxwell's equations

$$F_{ij;k} + F_{jk;i} + F_{ki;j} = 0$$

leads to $F_{23} = \text{const} = H$ (say). Hence

$$h_1 = AH/\bar{\mu}BC \tag{6}$$

The field equation for the line element (1) is

$$\frac{1}{A^2}\left(-\frac{B_{44}}{B} - \frac{C_{44}}{C} - \frac{B_4C_4}{BC} + \frac{A_4B_4}{AB} + \frac{A_4C_4}{AC}\right) = 8\pi\left(p - \frac{H^2}{2\bar{\mu}B^2C^2}\right) \tag{7}$$

$$\frac{1}{A^2}\left(-\frac{C_{44}}{C} - \frac{A_{44}}{A} + \frac{A_4^2}{A^2}\right) = 8\pi\left(p + \frac{H^2}{2\bar{\mu}B^2C^2}\right) \tag{8}$$

$$\frac{1}{A^2}\left(-\frac{B_{44}}{B} - \frac{A_{44}}{A} + \frac{A_4^2}{A^2}\right) = 8\pi\left(p + \frac{H^2}{2\bar{\mu}B^2C^2}\right) \tag{9}$$

$$\frac{1}{A^2}\left(\frac{A_4B_4}{AB} + \frac{A_4C_4}{AC} + \frac{B_4C_4}{BC}\right) = 8\pi\left(\varepsilon + \frac{H^2}{2\bar{\mu}B^2C^2}\right) \tag{10}$$

2. SOLUTION OF THE FIELD EQUATIONS

Equations (7)-(10) are four equations in five unknowns, A, B, C, ε, and p. For the complete determination of the set, we assume that the model is filled with a stiff fluid of perfect fluid distributions, so that we have $\varepsilon = p$, which leads to

$$\left(\frac{A_4}{A}\right)_4 + \frac{A_4}{A}\left(\frac{B_4}{B} + \frac{C_4}{C}\right) = -\left(\frac{B_{44}}{B} + \frac{B_4 C_4}{BC}\right) \tag{11}$$

From equations (7)-(9), we have

$$\left(\frac{A_4}{A}\right)_4 + \frac{A_4}{A}\left(\frac{B_4}{B} + \frac{C_4}{C}\right) = \frac{B_{44}}{B} + \frac{B_4 C_4}{BC} - \frac{8\pi H^2 A^2}{\bar{\mu} B^2 C^2} \tag{12}$$

and

$$\frac{B_{44}}{B} - \frac{C_{44}}{C} = 0 \tag{13}$$

Equation (11) leads to

$$\frac{A_4}{A} = -\frac{B_4}{B} + \frac{L}{BC} \tag{14}$$

where L is a constant of integration. From equations (11) and (12), we have

$$\frac{B_{44}}{B} + \frac{B_4 C_4}{BC} = \frac{4\pi H^2 A^2}{\bar{\mu} B^2 C^2} \tag{15}$$

which leads to

$$(B_4 C)_4 - \frac{K^2 A^2}{BC} = 0 \tag{16}$$

where

$$K^2 = 4\pi H^2 / \bar{\mu} \tag{17}$$

Putting $BC = \mu$ and $B/C = \nu$ in (13) and (16), we have

$$\mu_{44} + \left(\frac{\nu_4}{\nu}\mu\right)_4 - \frac{2K^2 A^2}{\mu} = 0 \tag{18}$$

and

$$\left(\frac{\nu_4}{\nu}\mu\right)_4 = 0 \tag{19}$$

Equation (19) leads to

$$\frac{\nu_4}{\nu}=\frac{k}{\mu} \tag{20}$$

where k is a constant of integration. Using (18) and (19), we have

$$\mu\mu_{44}=2K^2A^2 \tag{21}$$

From (21), we get

$$\frac{A_4}{A}=\frac{\mu_4}{2\mu}+\frac{\mu_{444}}{2\mu_{44}} \tag{22}$$

From equations (14), (20), and (22), we have

$$\frac{\mu\mu_{444}}{\mu_{44}}+2\mu_4=2\alpha \tag{23}$$

where

$$\alpha=L-k/2 \tag{24}$$

Putting $\mu_4=f(\mu)$ in (23), we have

$$\mu(ff''+f'^2)+2ff'=2\alpha f' \tag{25}$$

which leads to

$$\frac{\partial}{\partial\mu}\left(\mu ff'+\frac{f^2}{2}\right)=\frac{\partial}{\partial\mu}(2\alpha f) \tag{26}$$

which on integration leads to

$$\mu ff'+f^2/2=2\alpha f+\beta \tag{27}$$

where β is a constant of integration. From equation (27), we have

$$\mu=\frac{l(f-2\alpha+\gamma)^{2\alpha/\gamma-1}}{(f-2\alpha-\gamma)^{2\alpha/\gamma+1}} \tag{28}$$

where l is a constant of integration and $\gamma=(4\alpha^2+2\beta)^{1/2}$. From equations (20) and (28), we have

$$\nu=N\left(\frac{f-2\alpha+\gamma}{f-2\alpha-\gamma}\right)^{k/\gamma} \tag{29}$$

where N is a constant of integration. From (22), (28), and (29), we get

$$A=\frac{(\gamma+f-2\alpha)^{1/2}(\gamma-f+2\alpha)^{1/2}}{2K} \tag{30}$$

From (28) and (29), we have

$$B^2 = \mu\nu = lN\frac{(f-2\alpha+\gamma)^{(2\alpha+k)/\gamma-1}}{(f-2\alpha-\gamma)^{(2\alpha+k)/\gamma-1}} \tag{31}$$

and

$$C^2 = \frac{\mu}{\nu} = \frac{l}{N}\frac{(f-2\alpha+\gamma)^{(2\alpha-k)/\gamma-1}}{(f-2\alpha-\gamma)^{(2\alpha-k)/\gamma+1}} \tag{32}$$

Hence, the metric reduces to the form

$$\begin{aligned} ds^2 = {} & \frac{(\gamma+f-2\alpha)(\gamma-f+2\alpha)}{4K^2}\left(dx^2 - \frac{df^2}{f^2 f'^2}\right) \\ & + lN\frac{(f-2\alpha+\gamma)^{(2\alpha+k)/\gamma-1}}{(f-2\alpha-\gamma)^{(2\alpha+k)/\gamma+1}}\, dy^2 \\ & + \frac{l}{N}\frac{(f-2\alpha+\gamma)^{(2\alpha-k)/\gamma-1}}{(f-2\alpha-\gamma)^{(2\alpha-k)/\gamma+1}}\, dz^2 \end{aligned} \tag{33}$$

After suitable transformation of coordinates, the metric (33) reduces to the form

$$\begin{aligned} ds^2 = {} & \frac{\gamma^2-T^2}{4K^2}\left[dX^2 - 4\left(\frac{\gamma-T}{\gamma+T}\right)^{-4\alpha/\gamma}\frac{1}{(\gamma^2-T^2)^4}\, dT^2\right] \\ & + \left(\frac{\gamma-T}{\gamma+T}\right)^{-(2\alpha+k)/\gamma}\frac{1}{\gamma^2-T^2}\, dY^2 \\ & + \left(\frac{\gamma-T}{\gamma+T}\right)^{-(2\alpha-k)/\gamma}\frac{1}{\gamma^2-T^2}\, dZ^2 \end{aligned} \tag{34}$$

which, by the transformation $T = \gamma\cos 2K\tau$, reduces to the form

$$\begin{aligned} ds^2 = {} & \gamma^2\frac{\sin^2 2K\tau}{4K^2}\, dX^2 \frac{4l^2}{\gamma^2}\left(\frac{1-\cos 2K\tau}{1+\cos 2K\tau}\right)^{-4\alpha/\gamma}\frac{d\tau^2}{\sin^4 2K\tau} \\ & + \left(\frac{1-\cos 2K\tau}{1+\cos 2K\tau}\right)^{-(2\alpha+k)/\gamma}\frac{1}{\gamma^2\sin^2 2K\tau}\, dY^2 \\ & + \left(\frac{1-\cos 2K\tau}{1+\cos 2K\tau}\right)^{-(2\alpha-k)/\gamma}\frac{1}{\gamma^2\sin^2 2K\tau}\, dZ^2 \end{aligned} \tag{35}$$

When $K \to 0$ then the metric (35) reduces to the form

$$ds^2 = \tau^2\left(\gamma^2\, dX^2 - \frac{l^2}{4\gamma^2}\, d\tau^2\right) + dY^2 + dZ^2 \tag{36}$$

with $2\alpha + \gamma = 0$ and $k = 0$.

3. SOME PHYSICAL AND GEOMETRICAL FEATURES

The density for the model (34) is given by

$$8\pi\varepsilon = \frac{K^2}{l^2(\gamma^2 - T^2)} \frac{(T-\gamma)^{4\alpha/\gamma+2}}{(T+\gamma)^{4\alpha/\gamma-2}} [(4\alpha^2 - k^2 + l^2\gamma^2) - T^2(1+l^2)] \quad (37)$$

The model has to satisfy the reality conditions Ellis (1971)

(i) $\varepsilon + p > 0$
(ii) $\varepsilon + 3p > 0$

Conditions (i) and (ii) together lead to

$$T^2 < \frac{(4\alpha^2 - k^2 + l^2\gamma^2)}{(1+l^2)} \quad (38)$$

From (38) we find that

$$4\alpha^2 + l^2\gamma^2 > k^2 \quad (39)$$

The scalar of expansion θ calculated for the flow vector v^i is given by

$$\theta = \frac{K(T+4\alpha)}{(\gamma^2 - T^2)^{1/2}} \frac{(T-\gamma)^{2\alpha/\gamma+1}}{(T+\gamma)^{2\alpha/\gamma-1}} \quad (40)$$

The rotation w is identically zero and the shear is given by

$$\sigma^2 = \frac{K^2(T-\gamma)^{4\alpha/\gamma+2}}{3l^2(\gamma^2 - T^2)(T+\gamma)^{4\alpha/\gamma-2}} (4T^2 + 4\alpha^2 + 3k^2 + 8\alpha T) \quad (41)$$

The nonvanishing components of the conformal curvature tensor are

$$C^{12}_{12} = \frac{K^2(T-\gamma)^{4\alpha/\gamma+2}}{3l^2(\gamma^2 - T^2)(T+\gamma)^{4\alpha/\gamma-2}} \times (3T^2 + 4\alpha^2 - \gamma^2 - k^2 + 6\alpha T + kT) \quad (42)$$

$$C^{13}_{13} = \frac{K^2(T-\gamma)^{4\alpha/\gamma+2}}{3l^2(\gamma^2 - T^2)(T+\gamma)^{4\alpha/\gamma-2}} \times (3T^2 + 4\alpha^2 - \gamma^2 - k^2 + 6\alpha T - kT) \quad (43)$$

$$C^{23}_{23} = \frac{-2K^2(T-\gamma)^{4\alpha/\gamma+2}}{3l^2(\gamma^2 - T^2)(T+\gamma)^{4\alpha/\gamma-2}} \times (3T^2 + 4\alpha^2 - \gamma^2 - k^2 + 6\alpha T) \quad (44)$$

From (40), we find that the model will represent an expanding universe when $l < 0$. However, the expansion in the model stops when either $T = \gamma$ or $T = -4\alpha$. The effect of a magnetic field is to cause the model to expand up to a finite interval of time. The magnetic field introduces inhomogeneity in density. The model represents an expanding, shearing, nonrotating, and geodetic universe in general. The space-time is Petrov type D when $k=0$ and nondegenerate Petrov type I otherwise. Since $\lim_{T\to\infty}(\sigma/\theta)\neq 0$, the model does not approach isotropy for large values of T.

In the absence of a magnetic field, the space-time reduces to type D. Also, since $\lim_{T\to\infty}(\sigma/\theta)\neq 0$, the model does not approach isotropy for large values of T in this case also. When $K\to 0$, $\theta\to -\gamma/l\tau$, which shows that the expansion in the model stops for large values of T in the absence of a magnetic field.

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